Recitation 1. February 23

Focus: Rules of matrix multiplication, Gaussian and Gauss-Jordan elimination.

The most basic rule that you should remember: **row column**. It shows the order in which you write or compute:

- The first index denotes the row, the second number the column.
- You multiply a row by a column to get a number.
- An $m \times n$ matrix has m rows and n columns.

The formula **left matrix multiplication corresponds to row operations** explains the mathemagic behind Gaussian or Gauss-Jordan elimination. More precisely, performing row operations on a matrix A is the same as doing LA for some other matrix L, which turns out to be a products of elimination, diagonal and permutation matrices.

1. *Rules of matrix multiplication. (Section 2.4 of Strang.)* Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Which of these matrix operations are allowed?

a) AB

b) $(A+B)C$

c) $C(A+B)$

d) AD

e) DA

f) CAD

Solution: In order to multiply two matrices, number of columns in the first should be equal to number of rows in the second.

a) AB not allowed: we cannot multiply a 2×3 matrix by a 2×3 matrix.

b) $(A+B)C$ not allowed, for the same reason as above.

c) $C(A+B) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 \\ -4 & -4 & -8 \end{bmatrix}$.

d) $AD = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$.

e) DA not allowed, for the same reason as above.

f) $CAD = C(AD) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} -14 \\ -18 \end{bmatrix}$.

2. *Binomial formula for matrices.* Show that $(A+B)^2$ is different from $A^2+2AB+B^2$ when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

Write down the correct rule: $(A+B)^2 = A^2 + \cdots + B^2$.

Solution:

- $(A+B)^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}$;
- $A^2+2AB+B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.
- The correct rule is $(A+B)^2 = A^2 + AB + BA + B^2$.

3. Consider the following matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

How is each row of BA, CA, DA related to the rows of A?

Solution:

- The first row of BA is twice the first row of A, and the second is minus the second row of A.
- The first row of CA is the second row of A, while the second row is zero.
- The first row of DA is the second row of A and the second row of DA is minus the second row of A.

So you can see that multiplying a matrix A by another matrix on the left performs row operations. Similarly, multiplying a matrix A by another matrix on the right performs column operations.

4. *Gaussian elimination (row echelon form).* Solve the following system of linear equations by Gaussian elimination:

$$\begin{cases} x + 2y + 3z = 1 \\ \quad\;\; y + \;\; z = 2 \\ 3x + \;\; y - \;\; z = 3 \end{cases}$$

Solution: We start by writing the system of linear equations in the matrix form:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Now we perform Gaussian elimination on the augmented matrix ($r_i + \alpha r_j$ means that we add α times the j–th row to the i–th row):

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 3 & 1 & -1 & 3 \end{array}\right] \xrightarrow{r_3 - 3r_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & -5 & -10 & 0 \end{array}\right] \xrightarrow{r_3 + 5r_2} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & \boxed{-5} & 10 \end{array}\right]$$

The system of equations is now equivalent to:

$$\begin{cases} x + 2y + 3z = 1 \\ \quad\;\; y + \;\; z = 2 \\ \quad\quad\;\; - 5z = 10 \end{cases}$$

We can solve this by back substitution, i.e. first get $z = -2$ from the last equation, the plug this into the second equation and get $y = 4$, then plug this into the first equation and get $x = -1$.

5. *Gauss-Jordan elimination (reduced row echelon form).* Same problem as above, but do Gauss-Jordan elimination.

Solution: Starting from the row echelon form we already computed above, let's continue the process of elimination until we get the reduced row echelon form ($r_i \cdot \alpha$ means that we multiply the i–th row by the number α):

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -5 & 10 \end{array}\right] \xrightarrow{r_3 \cdot \left(-\frac{1}{5}\right)} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{r_1 - 2r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{r_1 - r_3}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{r_2 - r_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & \boxed{1} & -2 \end{array}\right]$$

The matrix above is now in reduced row echelon form because not only are all pivots to the right of the pivots above them, but all pivots are 1 and all entries above a pivot are 0. The system of equations is now equivalent to:

$$\begin{cases} x = -1 \\ y = 4 \\ z = -2 \end{cases}$$

so we get the same solution as before. Upshot: while Gauss-Jordan elimination (a.k.a. reduced row echelon form) takes more steps than Gaussian elimination (a.k.a. row echelon form), it produces a much simpler system of equations, whose solution is much more obvious.